Multivariate Chain Rule

Goal: Understand derivatives of composite multivariate functions.

Setup: Let $f : D_0 \subseteq \mathbb{R}^n \to \mathbb{R}$ and suppose

$$x_1, x_2, \cdots, x_n : D_1 \subseteq \mathbb{R}^m \to \mathbb{R}.$$

We think of f as a function $f(x_1, x_2, \cdots, x_n)$ and

$$x_1 = x_1(t_1, t_2, \cdots, t_m)$$
$$x_2 = x_2(t_1, t_2, \cdots, t_m)$$
$$\vdots$$
$$x_n = x_n(t_1, t_2, \cdots, t_m)$$

Thus we can form the composite function

$$g(t_1, t_2, \cdots, t_m) = f(x_1(t_1, t_2, \cdots, t_m), x_2(t_1, t_2, \cdots, t_m), \cdots, x_n(t_1, t_2, \cdots, t_m))$$

Q: How can we compute the partial derivatives $\frac{\partial g}{\partial t_k}$ using the partial derivatives of f and $x_1, x_2, \cdots, x_n$?

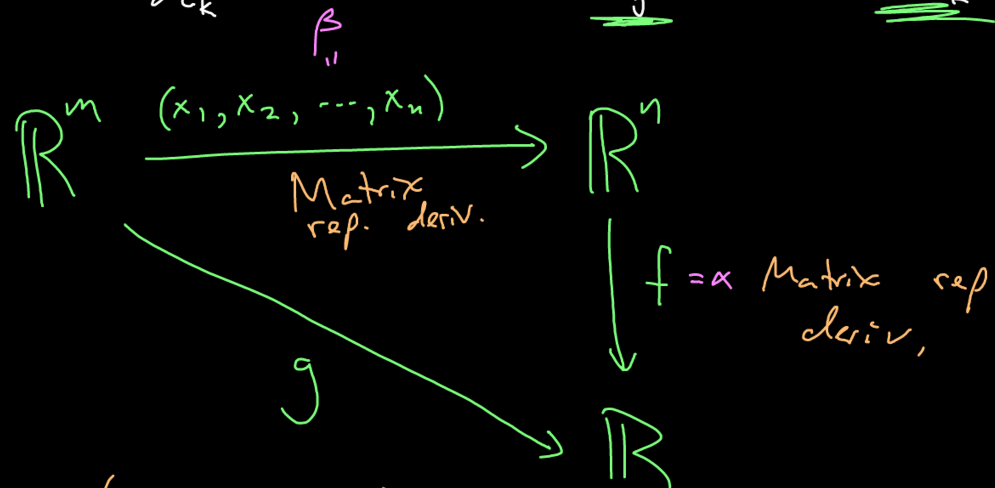
NB: I will give you an answer to this question with a correct, formal, but non-rigorous description of where the formula comes from. Unfortunately, the proof of this result would take us too far afield...

Description of Multivariate Chain Rule with Pictures

Have: $g(t_1, t_2, \dots, t_m) = f(x_1(t_1, t_2, \dots, t_m), x_2(t_1, t_2, \dots, t_m), \dots, x_n(t_1, t_2, \dots, t_m))$

Want: Formula for $\frac{\partial g}{\partial t_k}$ in terms of $\frac{\partial f}{\partial x_j}$'s and $\frac{\partial x_j}{\partial t_k}$'s ...

Picture:



$\text{deriv}(\alpha \circ \beta) = (\text{deriv}(\alpha) \circ \beta) \cdot \text{deriv}(\beta)$ ← Calculus I

$\text{deriv}(g) = \text{deriv}(f \circ (x_1, \dots, x_n))$ matrix product ↙ Calculus III

$= (\text{deriv}(f) \circ (x_1, \dots, x_n)) \cdot \text{deriv}((x_1, \dots, x_n))$

Remark: Matrix products from linear algebra have formula:

$$A \cdot B = [a_{i,j}]_{i,j=1}^{t,s} \cdot [b_{j,k}]_{j,k=1}^{s,r} = \left[\sum_{j=1}^{s} a_{i,j} \cdot b_{j,k}\right]_{i,k=1}^{t,r}$$

Point: Using the Calculus I mentality and knowledge of Linear Algebra, we plausibly obtain the formula

$$\frac{\partial f}{\partial t_k} = \begin{pmatrix} \text{derivative of} \\ f \text{ w.r.t. } \vec{x} \end{pmatrix} \cdot \begin{pmatrix} \text{derivative of} \\ \vec{x} \text{ w.r.t. } t_k \end{pmatrix}$$

(matrix product)

$$\left(\text{So } \frac{\partial f}{\partial t_k} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_k} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_k} + \dots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_k} \right) \star$$

↑ Multivariate Chain Rule!!!

Ex: Compute $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ for $f(x,y) = e^x \sin(y)$ where $x(s,t) = st^2$ and $y(s,t) = s^2 t$.

Sol 1 (Composition first): $f(x,y) = f(x(s,t), y(s,t)) = e^{x(s,t)} \sin(y(s,t)) = e^{st^2} \sin(s^2 t)$

$$\frac{\partial f}{\partial s} = \frac{\partial}{\partial s}\left[e^{st^2} \sin(s^2 t)\right] = e^{st^2} \frac{\partial}{\partial s}\left[\sin(s^2 t)\right] + \frac{\partial}{\partial s}\left[e^{st^2}\right] \sin(s^2 t)$$

$$= e^{st^2} \cdot \cos(s^2 t) \cdot 2st + e^{st^2} \cdot t^2 \cdot \sin(s^2 t) = te^{st^2}\left(2s \cos(s^2 t) + t \sin(s^2 t)\right)$$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left[e^{st^2} \sin(s^2 t)\right] = e^{st^2} \frac{\partial}{\partial t}\left[\sin(s^2 t)\right] + \frac{\partial}{\partial t}\left[e^{st^2}\right] \sin(s^2 t)$$

$$= e^{st^2} \cdot \cos(s^2 t) \cdot s^2 + e^{st^2} \cdot 2st \cdot \sin(s^2 t) = se^{st^2}\left(s \cos(s^2 t) + 2t \sin(s^2 t)\right)$$

Sol 2 (chain rule):

$$\frac{\partial f}{\partial x} = e^x \sin(y) \qquad \frac{\partial x}{\partial s} = t^2 \qquad \frac{\partial x}{\partial t} = 2st$$

$$\frac{\partial f}{\partial y} = e^x \cos(y) \qquad \frac{\partial y}{\partial s} = 2st \qquad \frac{\partial y}{\partial t} = s^2$$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= e^{x(s,t)} \sin(y(s,t)) \cdot t^2 + e^{x(s,t)} \cos(y(s,t)) \cdot 2st$$

$$= e^{st^2} \sin(s^2 t) t^2 + e^{st^2} \cos(s^2 t) \cdot 2st$$

$$= te^{st^2}\left(t \sin(s^2 t) + 2s \cos(s^2 t)\right)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= e^{x(s,t)} \sin(y(s,t)) \cdot 2st + e^{x(s,t)} \cos(y(s,t)) \cdot s^2$$

$$= e^{st^2} \sin(s^2 t) \cdot 2st + e^{st^2} \cos(s^2 t) \cdot s^2$$

$$= se^{st^2}\left(2t \sin(s^2 t) + s \cos(s^2 t)\right)$$

Ex: Compute $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$ for $f(x,y) = x^2 + xy - y^2$

where $x = r\cos(\theta)$ and $y = r \sin(\theta)$.

Sol:

$$\frac{\partial f}{\partial x} = 2x + y = 2r\cos(\theta) + r\sin(\theta) \qquad \frac{\partial x}{\partial r} = \cos(\theta) \qquad \frac{\partial x}{\partial \theta} = -r\sin(\theta)$$

$$\frac{\partial f}{\partial y} = x - 2y = r\cos(\theta) - 2r\sin(\theta) \qquad \frac{\partial y}{\partial r} = \sin(\theta) \qquad \frac{\partial y}{\partial \theta} = r\cos(\theta)$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \left(2r\cos(\theta) + r\sin(\theta)\right) \cdot \cos(\theta) + \left(r\cos(\theta) - 2r\sin(\theta)\right) \sin(\theta)$$

$$= r\left(2\cos^2(\theta) + 2\sin(\theta)\cos(\theta) - 2\sin^2(\theta)\right)$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= r\left(2\cos(\theta) + \sin(\theta)\right) \cdot -r\sin(\theta) + r\left(\cos(\theta) - 2\sin(\theta)\right) \cdot r\cos(\theta)$$

$$= r^2\left(\cos^2(\theta) - 4\sin(\theta)\cos(\theta) - \sin^2(\theta)\right)$$

Ex: Compute $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial \theta}$ for $w = xy + yz + zx$

where $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = r\theta$.

Sol:

$$\frac{\partial w}{\partial x} = y + z = r(\sin(\theta) + \theta) \qquad \frac{\partial x}{\partial r} = \cos(\theta) \qquad \frac{\partial y}{\partial r} = \sin(\theta) \qquad \frac{\partial z}{\partial r} = \theta$$

$$\frac{\partial w}{\partial y} = x + z = r(\cos(\theta) + \theta) \qquad \frac{\partial x}{\partial \theta} = -r\sin(\theta) \qquad \frac{\partial y}{\partial \theta} = r\cos(\theta) \qquad \frac{\partial z}{\partial \theta} = r$$

$$\frac{\partial w}{\partial z} = y + x = r(\sin(\theta) + \cos(\theta))$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= r(\sin(\theta)+\theta)\cdot\cos(\theta) + r(\cos(\theta)+\theta)\cdot\sin(\theta) + r(\sin(\theta)+\cos(\theta))\cdot\theta$$

$$= 2r\left(\sin(\theta)\cos(\theta) + \theta(\cos(\theta)+\sin(\theta))\right)$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial \theta}$$

$$= r(\sin(\theta)+\theta)\cdot -r\sin(\theta) + r(\cos(\theta)+\theta)\cdot r\cos(\theta) + r(\sin(\theta)+\cos(\theta))\cdot r$$

$$= r^2\left(\cos^2(\theta) - \sin^2(\theta) + \theta(\cos(\theta)-\sin(\theta)) + \sin(\theta) + \cos(\theta)\right)$$

Ex: Compute $\frac{\partial T}{\partial p}$, $\frac{\partial T}{\partial q}$, and $\frac{\partial T}{\partial r}$ for $T = \frac{v}{2u+v}$ where

$u = pq\sqrt{r}$ and $v = p\sqrt{q}\,r$ at $(p,q,r) = (2,1,4)$.

In the video, Chris forgot to finish computing! All that remains is evaluation...

Sol: $T(u,v) = v(2u+v)^{-1}$

$$\frac{\partial T}{\partial u} = -2v(2u+v)^{-2}$$

$$= -2pq^{1/2}r\left(2pqr^{1/2} + pq^{1/2}r\right)^{-2}$$

$$= -\frac{2pq^{1/2}r}{p^2qr\left(2q^{1/2}+r^{1/2}\right)^2}$$

$$= -\frac{2}{pq^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2} = \frac{-2}{2\cdot\sqrt{1}\left(2\sqrt{1}+\sqrt{4}\right)^2} = -\frac{1}{16}$$

$$\frac{\partial T}{\partial v} = v\cdot -(2u+v)^{-2} + (2u+v)^{-1}$$

$$= \left(-v + (2u+v)\right)(2u+v)^{-2}$$

$$= \frac{2pqr^{1/2}}{\left(2pqr^{1/2} + pq^{1/2}r\right)^2}$$

$$= \frac{2pqr^{1/2}}{p^2qr\left(2q^{1/2}+r^{1/2}\right)^2}$$

$$= \frac{2}{pr^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2} = \frac{2}{2\cdot\sqrt{4}\left(2\sqrt{1}+\sqrt{4}\right)^2} = \frac{1}{32}$$

$$\frac{\partial u}{\partial p} = qr^{1/2} = 1\cdot\sqrt{4} = 2 \qquad \frac{\partial u}{\partial q} = pr^{1/2} = 2\cdot\sqrt{4} = 4 \qquad \frac{\partial u}{\partial r} = \frac{1}{2}pqr^{-1/2} = \frac{1}{2}\cdot 2\cdot 1\cdot 4^{-1/2} = \frac{1}{2}$$

$$\frac{\partial v}{\partial p} = q^{1/2}r = \sqrt{1}\cdot 4 = 4 \qquad \frac{\partial v}{\partial q} = \frac{1}{2}pq^{-1/2}r = \frac{1}{2}\cdot 2\cdot 1^{-1/2}\cdot 4 = 4 \qquad \frac{\partial v}{\partial r} = pq^{1/2} = 2\cdot\sqrt{1} = 2$$

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial u}\cdot\frac{\partial u}{\partial p} + \frac{\partial T}{\partial v}\cdot\frac{\partial v}{\partial p} = -\frac{2qr^{1/2}}{pq^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2} + \frac{2q^{1/2}r}{pr^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2} = \frac{-2\sqrt{qr} + 2\sqrt{qr}}{p\left(2q^{1/2}+r^{1/2}\right)^2} = 0.$$

$$= -\frac{1}{16}\cdot 2 + \frac{1}{32}\cdot 4 = 0$$

$$\frac{\partial T}{\partial q} = \frac{\partial T}{\partial u}\cdot\frac{\partial u}{\partial q} + \frac{\partial T}{\partial v}\cdot\frac{\partial v}{\partial q} = -\frac{2pr^{1/2}}{pq^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2} + \frac{2\cdot\frac{1}{2}pq^{-1/2}r}{pr^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2} = \frac{q^{-1/2}r^{1/2} - 2q^{-1/2}r^{1/2}}{\left(2q^{1/2}+r^{1/2}\right)^2} = -\frac{r^{1/2}}{q^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2}$$

$$= -\frac{1}{16}\cdot 4 + \frac{1}{32}\cdot 4 = -\frac{1}{8}$$

$$\frac{\partial T}{\partial r} = \frac{\partial T}{\partial u}\cdot\frac{\partial u}{\partial r} + \frac{\partial T}{\partial v}\cdot\frac{\partial v}{\partial r} = -\frac{2\cdot\frac{1}{2}pqr^{-1/2}}{pq^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2} + \frac{2\cdot pq^{1/2}}{pr^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2} = \frac{-q^{1/2}r^{-1/2} + 2q^{1/2}r^{-1/2}}{\left(2q^{1/2}+r^{1/2}\right)^2} = \frac{q^{1/2}}{r^{1/2}\left(2q^{1/2}+r^{1/2}\right)^2}$$

$$= -\frac{1}{16}\cdot\frac{1}{2} + \frac{1}{32}\cdot 2 = \frac{1}{32}$$

Ex: Show $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$ for $g(s,t) = f(s^2 - t^2, t^2 - s^2)$.

Sol: First compute $\frac{\partial g}{\partial s}$ and $\frac{\partial g}{\partial t}$:

$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}2s + \frac{\partial f}{\partial y}\cdot -2s = 2s\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right)$$

$$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}\cdot -2t + \frac{\partial f}{\partial y}\cdot 2t = 2t\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right)$$

(**NB**: We computed the above using the chain rule).

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = t\cdot 2s\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) + s\cdot 2t\left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right)$$

$$= 2st\left(\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) + \left(\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\right)\right)$$

$$= 2st\cdot 0 = 0.$$

$\blacksquare$

Implicit Function Theorem

Idea: Sometimes functions are given to us implicitly via relations between variables, rather than explicitly.

Goal: Take derivatives of implicitly defined multivariate functions.

☆ **Prop (Implicit Function Theorem):** Let $F : B \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$ where B is a ball and let $p \in B$ with $F(p) = 0$. If $\left.\frac{\partial F}{\partial x_{n+1}}\right|_p \neq 0$ and $\frac{\partial F}{\partial x_k}$ is continuous on B, then locally $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ and $\frac{\partial f}{\partial x_k} = -\frac{\partial F}{\partial x_k} \Big/ \frac{\partial F}{\partial x_{n+1}}$ at p.

pf: Applying the chain rule to $F = 0$ we obtain

☆ $$0 = \frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial x_k} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_k} + \cdots + \frac{\partial F}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_k} + \frac{\partial F}{\partial x_{n+1}} \cdot \frac{\partial x_{n+1}}{\partial x_k}.$$

Now notice $x_{n+1} = f$ and $\frac{\partial x_j}{\partial x_k} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$, so this equation reduces to $0 = \frac{\partial F}{\partial x_k} + \frac{\partial F}{\partial x_{n+1}} \cdot \frac{\partial f}{\partial x_k}$. Solving for $\frac{\partial f}{\partial x_k} = -\frac{\partial F}{\partial x_k} \Big/ \frac{\partial F}{\partial x_{n+1}}$ yields the result. ▨

NB: This result justifies our "implicit differentiation" technique from Calculus I, which is often used in a variety of applications, including physics and other fields involving related rates.

Ex: Compute $\frac{dy}{dx}$ for $y(x)$ defined implicitly by $\arctan(x^2y) = x + xy^2$.

Sol: Apply the Implicit Function Theorem to

$$F(x,y) = \arctan(x^2y) - x - xy^2.$$

$$\frac{\partial F}{\partial x} = \frac{1}{1+(x^2y)^2}\cdot 2xy - 1 - y^2 \qquad \frac{\partial F}{\partial y} = \frac{1}{1+(x^2y)^2}\cdot x^2 - 2xy$$

$$\therefore \frac{dy}{dx} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y}$$

$$= -\frac{2xy - (1+y^2)(1+x^4y^2)}{1+x^4y^2} \Big/ \frac{x^2 - 2xy(1+x^4y^2)}{1+x^4y^2} = -\frac{2xy - (1+y^2)(1+x^4y^2)}{x^2 - 2xy(1+x^4y^2)}$$

∎

Ex: Compute the partial derivatives of $z(x,y)$ defined by $x^3 + y^3 + z^3 + 6xyz = 1$.

Sol: Apply the Implicit Function Theorem to $F(x,y,z) = x^3 + y^3 + z^3 + 6xyz - 1$

$$\frac{\partial F}{\partial x} = 3x^2 + 6yz \qquad \frac{\partial F}{\partial y} = 3y^2 + 6xz \qquad \frac{\partial F}{\partial z} = 3z^2 + 6xy$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

∎

Ex: Compute the partial derivatives of $z(x,y)$ defined by $e^z = xyz$.

Sol: Apply Implicit Function Theorem to $F(x,y,z) = e^z - xyz$.

$$\frac{\partial F}{\partial x} = -yz \qquad \frac{\partial F}{\partial y} = -xz \qquad \frac{\partial F}{\partial z} = e^z - xy.$$

$$\therefore \frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial z} = -\frac{-yz}{e^z - xy} = \frac{yz}{e^z - xy}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y} \Big/ \frac{\partial F}{\partial z} = -\frac{-xz}{e^z - xy} = \frac{xz}{e^z - xy}.$$

∎

Gradient as Derivative

At this point, we can compute the instantaneous rate of change of a function in any given direction.

Q: Is this the derivative?

The derivative is fundamentally an object to approximate a function locally by a linear object (e.g. a line in Calculus I).
In this light, $D_{\vec{u}} f$ is somewhat of a failure... It does local approximation, but not in all directions simultaneously...

Defn: The gradient of $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ at $p \in D$ is

$$\vec{\nabla} f(p) = \left\langle \frac{\partial f}{\partial x_1}\Big|_p, \frac{\partial f}{\partial x_2}\Big|_p, \ldots, \frac{\partial f}{\partial x_n}\Big|_p \right\rangle.$$

NB: The gradient is the matrix of partial derivatives of f!

Ex: Compute the gradient of $f(x,y,z) = \cos(x^2+y^2)\arctan(z)$.

Sol:
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \left\langle \frac{\partial}{\partial x}\left[\cos(x^2+y^2)\arctan(z)\right], \frac{\partial}{\partial y}\left[\cos(x^2+y^2)\arctan(z)\right], \frac{\partial}{\partial z}\left[\cos(x^2+y^2)\arctan(z)\right] \right\rangle$$

$$= \left\langle -2x\sin(x^2+y^2)\arctan(z), -2y\sin(x^2+y^2)\arctan(z), \frac{\cos(x^2+y^2)}{1+z^2} \right\rangle$$

$\boxed{\text{KS}}$

Ex: Compute the gradient of $f(x,y,z) = x^2 + 3xyz - y^3z$.

Sol: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$$= \langle 2x + 3yz,\ 3xz - 3y^2z,\ 3xy - y^3 \rangle \quad \blacksquare$$

Ex: Compute the gradient of $f(x,y) = 7x\cos(x^2y)$.

Sol: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

$$= \langle 7x \cdot -\sin(x^2y)\cdot 2xy + 7\cdot\cos(x^2y),\ 7x \cdot -\sin(x^2y)\cdot x^2 \rangle$$

$$= 7\langle \cos(x^2y) - 2x^2y\sin(x^2y),\ -x^3\sin(x^2y) \rangle. \quad \blacksquare$$

Ex: Compute the gradient of $E(p,m,c) = \sqrt{p^2c^2 + m^2c^4}$.

Sol: $\nabla E = \left\langle \frac{\partial E}{\partial p}, \frac{\partial E}{\partial m}, \frac{\partial E}{\partial c} \right\rangle$

$$= \left\langle \tfrac{1}{2}(p^2c^2+m^2c^4)^{-1/2}\cdot 2pc^2,\ \tfrac{1}{2}(p^2c^2+m^2c^4)^{-1/2}\cdot 2mc^4,\ \tfrac{1}{2}(p^2c^2+m^2c^4)^{-1/2}(2p^2c + 4m^2c^3) \right\rangle$$

$$= \frac{c}{\sqrt{p^2c^2+m^2c^4}}\langle pc,\ mc^3,\ p^2 + 2m^2c^2 \rangle \quad \blacksquare$$

Properties of the Gradient

Prop: Let $f, g : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and $\alpha : \mathbb{R} \to \mathbb{R}$.

① $\nabla(f+g) = (\nabla f) + (\nabla g)$

② $\nabla(f \cdot g) = (\nabla f) \cdot g + f \cdot \nabla g$

③ $\nabla(\alpha \circ f) = (\alpha' \circ f) \cdot \nabla f$

Prop: Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and $p \in D$. For all unit vectors $\vec{u} \in \mathbb{R}^n$ we have $D_{\vec{u}} f(p) = \nabla f(p) \cdot \vec{u}$.

pf: Let $g(h) := f(\vec{p} + h\vec{u})$. We compute $g'(0)$ in two ways.

① By definition, $g'(0) = \lim_{h \to 0} \frac{f(\vec{p} + h\vec{u}) - f(\vec{p})}{h} = D_{\vec{u}} f(p)$.

② By the Chain Rule, $g'(0) = \left(\frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial h} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial h} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial h} \right) \Big|_{h=0,\, \vec{x}=\vec{p}}$

Now $x_i(h) = p_i + u_i h$ yields $\frac{dx_i}{dh} = u_i$ ③, so we have shown

$$D_{\vec{u}} f(p) \overset{①}{=} g'(0)$$

$$\overset{②}{=} \left(\frac{\partial f}{\partial x_1} \cdot \frac{\partial x_1}{\partial h} + \frac{\partial f}{\partial x_2} \cdot \frac{\partial x_2}{\partial h} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{\partial x_n}{\partial h} \right) \Big|_{h=0,\, \vec{x}=\vec{p}}$$

$$\overset{③}{=} \left(\frac{\partial f}{\partial x_1} u_1 + \frac{\partial f}{\partial x_2} u_2 + \cdots + \frac{\partial f}{\partial x_n} u_n \right) \Big|_{\vec{x}=\vec{p}}$$

$$= \nabla f(p) \cdot \vec{u}$$

∎

Ex: Compute the directional derivative $D_{\vec{u}}f(p)$ for ...

① $f(x,y) = \frac{x}{x^2+y^2}$, $\vec{u} = \langle 3,5\rangle$, $p = (1,2)$

Sol: $f(x,y) = x(x^2+y^2)^{-1}$

$$\nabla f = x\cdot\nabla\left((x^2+y^2)^{-1}\right) + (x^2+y^2)^{-1}\nabla x$$

$$= x\cdot -(x^2+y^2)^{-2}\,\nabla(x^2+y^2) + (x^2+y^2)^{-2}\cdot(x^2+y^2)\,\nabla x$$

$$= (x^2+y^2)^{-2}\left(-x\langle 2x,2y\rangle + (x^2+y^2)\langle 1,0\rangle\right)$$

$$= (x^2+y^2)^{-2}\langle y^2-x^2, -2xy\rangle$$

$$\nabla f(1,2) = \frac{1}{(1^2+2^2)^2}\langle 2^2-1^2, -2\cdot1\cdot2\rangle = \frac{1}{25}\langle 3,-4\rangle$$

$$D_{\vec{u}}f(p) = \nabla f(p)\cdot\frac{\vec{u}}{|\vec{u}|}$$

$$= \frac{1}{25}\langle 3,-4\rangle\cdot\frac{1}{\sqrt{3^2+5^2}}\langle 3,5\rangle$$

$$= \frac{1}{25}\cdot\frac{1}{\sqrt{34}}(9-20)$$

$$= -\frac{11}{25\sqrt{34}}$$

② $f(x,y,z) = xy^2\arctan(z)$, $\vec{u} = \langle -2,1,1\rangle$, $p = (1,-1,1)$

Sol: $D_{\vec{u}}f(p) = \nabla f(p)\cdot\frac{\vec{u}}{|\vec{u}|}$

$$\nabla f = \left\langle y^2\arctan(z),\ 2xy\arctan(z),\ \frac{xy^2}{1+z^2}\right\rangle$$

$$\nabla f(p) = \left\langle (-1)^2\arctan(1),\ 2\cdot1\cdot(-1)\arctan(1),\ \frac{1\cdot(-1)^2}{1+1^2}\right\rangle$$

$$= \left\langle \frac{\pi}{4}, -\frac{\pi}{2}, \frac{1}{2}\right\rangle$$

$$\therefore D_{\vec{u}}f(p) = \nabla f(p)\cdot\frac{\vec{u}}{|\vec{u}|} = \left\langle \frac{\pi}{4}, -\frac{\pi}{2}, \frac{1}{2}\right\rangle\cdot\frac{1}{\sqrt{2^2+1^2+1^2}}\langle -2,1,1\rangle$$

$$= \frac{1}{\sqrt{6}}\left(\frac{\pi}{4}\cdot -2 + -\frac{\pi}{2}\cdot 1 + \frac{1}{2}\right) = \frac{1}{\sqrt{6}}\left(\frac{1}{2}-\pi\right)$$

③ $f(r,s,t) = \ln(3r+6s+9t)$, $\vec{u} = \langle 4,12,6\rangle$, $p = (1,1,1)$

Sol: $D_{\vec{u}}f(p) = \nabla f(p)\cdot\frac{\vec{u}}{|\vec{u}|}$

$$\nabla f = \frac{1}{3r+6s+9t}\nabla(3r+6s+9t) = \frac{1}{r+2s+3t}\langle 1,2,3\rangle$$

$$\nabla f(1,1,1) = \frac{1}{1+2+3}\langle 1,2,3\rangle = \frac{1}{6}\langle 1,2,3\rangle$$

$$D_{\vec{u}}f(p) = \nabla f(p)\cdot\frac{\vec{u}}{|\vec{u}|} = \frac{1}{6}\langle 1,2,3\rangle\cdot\frac{2}{\sqrt{4^2+12^2+6^2}}\langle 2,6,3\rangle$$

$$= \frac{1}{6}\cdot\frac{2}{2\sqrt{2^2+6^2+3^2}}(2+12+9) = \frac{23}{6\sqrt{49}} = \frac{23}{42}$$